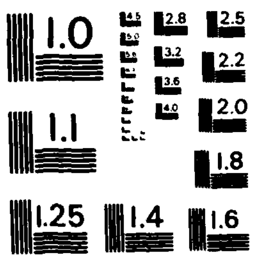

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A LOWER BOUND FOR THE BAYES RISK FOR TESTING SEQUENTIALLY THE SIGN OF THE DRIFT PARAMETER OF A WIENER PROCESS

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TECHNICAL REPORT NO. ONR 26

MARCH 1983

PREPARED UNDER CONTRACT

N00014-75-C-0555 (NR-609-001)

FOR THE OFFICE OF NAVAL RESEARCH

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ABSTRACT

Let $x(t)$ be a Wiener process with drift μ and variance 1 per unit of time. For testing $H: \mu \le 0$ vs $A: \mu > 0$ with the loss function $|\mu|$ if the wrong decision is made and 0 otherwise, c cost of observation per unit time and μ has a prior distribution which is normal with mean 0 and variance σ_0^2, we followed an idea of Bickel and Yahav to obtain a lower bound for the Bayes risk and showed that this lower bound is strict as $\sigma_0 \to \infty$ for all c.

Key Words: Sequential tests, S.P.R.T, Bayes, stopping times, lower bound, asymptotic expansion.

AMS 1980 Subject Classification: Primary 62L10; Secondary 62C10

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1. Introduction : Let $x(t)$ be a Wiener process with drift μ and variance 1 per unit of time. Chernoff [2] considered the following problem, test

$$H: \mu \le 0 \quad \text{vs} \quad A: \mu > 0$$

with the loss function $|\mu|$ if the wrong decision is made and 0 otherwise, c cost of observation per unit time and μ has a prior distribution which is normal with mean μ_0 and variance σ_0^2. Chernoff [3] showed that the Bayes risk

$$(1.1) \quad B(\mu_0,\sigma_0^2) = c^{\frac{2}{3}}\left[K\sigma_0^{-1}\phi\left(\frac{\mu_0}{\sigma_0}\right) - 6c^{\frac{1}{3}}\sigma_0^{-2}\ln\sigma_0\left(1+o(1)\right)\right]$$

$$\text{as } \sigma_0 \to \infty$$

Where K is an unknown constant. Throughout this paper ϕ and Φ are the standard normal density and cumulative respectively.

By considering the above testing problem with the additional information of the magnitude of μ, Bickel and Yahav [1] obtained a lower bound for the Bayes risk for the case of μ having the improper prior distribution and conjectured that the lower bound can be attained as $c \downarrow 0$. In this note we assume that μ has a normal prior distribution with mean 0 and variance σ_0^2. By using similar techniques as in Bickel and Yahav [1], we obtained a lower bound for the Bayes risk, then showed that this lower bound is not asymptotically achievable as $\sigma_0^2 \to \infty$ for all $c > 0$.

2. __Lower Bound For Bayes Risk:__ From Chernoff [3], the posterior cost of wrong decision is given by

$$(2.1) \quad Y_t = (t+\sigma_0^{-2})^{-1/2} \left\{\phi(\alpha)-|\alpha|\Phi(-|\alpha|)\right\}$$

where $\alpha = (t+\sigma_0^{-2})^{-1/2} X(t)$. Let the posterior risk at time t be,

$$(2.2) \quad R(c,t) = Y_t + ct$$

We are interested in a stopping rule τ_0 for which

$$E[R(c,\tau_0)] = \inf_{\tau\epsilon T} E[R(c,\tau)]$$

where T is the class of all stopping times.

Using the idea of Bickel and Yahav [1], let us consider the following problem of testing,

$$H: \mu = \mu_0 \quad vs \quad A: \mu = -\mu_0$$

with $|\mu_0|$ for cost of wrong decision and prior distribution $P(\mu = \mu_0) = P(\mu = -\mu_0) = \frac{1}{2}$. Then the posterior cost of wrong decision is

$$\tilde{Y}_t = |\mu_0| \, P(X(t)\mu < 0 \mid X(t))$$

Let

$$\tilde{R}(c,t) = \tilde{Y}_t + ct$$

To solve the above Bayes problem, we have to find a stopping rule τ^* such that

$$E(\tilde{R}(c,\tau^*)) = \inf_{\tau\epsilon T} E(\tilde{R}(c,\tau))$$

From the property of S.P.R.T we have the following lemma.

<u>Lemma 2.1:</u> <u>The stopping rule</u> τ^*: <u>Stop the first</u> $|X(t)| = a$, <u>where</u> "a" <u>is determined by the minimization of</u>

$$|\mu_0|(1+\exp(2a|\mu_0|))^{-1} + ca|\mu_0|^{-1}(1-2(1+\exp(2a|\mu_0|))^{-1})$$

<u>is the optimal stopping rule for the above problem.</u>

<u>Lemma 2.2:</u>

$$(2\pi\sigma_0^2)^{-1/2} \int_{-\infty}^{\infty} E_\mu[\tilde{R}(c,\tau^*)]\exp(-\mu^2/2\sigma_0^2)d\mu \le E[R(c,\tau_0)]$$

<u>Proof:</u> τ_0 is a Bayes rule for a symmetric problem and hence is symmetric in μ. Hence

$$E_\mu[R(c,\tau_0)] \ge E_\mu(\tilde{R}(c,\tau^*)] \quad \text{for all } \mu$$

From it the lemma follows.

<u>Theorem:</u>

$$(2\pi\sigma_0^2)^{-1/2} \int_{-\infty}^{\infty} E_\mu[\tilde{R}(c,\tau^*)]\exp(-\mu^2/2\sigma_0^2)d\mu$$

$$= c^{\frac{2}{3}}[K'\sigma_0^{-1} - \tfrac{3}{2} c^{\frac{1}{3}}\sigma_0^{-2}\ln\sigma_0(1+o(1))]$$

$$\text{as } \sigma_0 \to \infty$$

<u>where</u>

$$K' = (2\pi)^{-\frac{1}{2}} 2^{\frac{1}{3}} 3^{-1} \int_1^{\infty} (z-z^{-1}+2\ln z)^{-4/3}(1+z^{-2}+2z^{-1})$$

$$(1+\ln z-z^{-1})dz$$

<u>Proof:</u> Let

$$(2.3) \qquad z = e^{2a\mu}$$

where "a" is the solution of the minimization problem in Lemma 2.1. Then z should satisfy the relation

$$(2.4) \qquad 2\mu^3 = c(z-z^{-1} + 2\ln z)$$

We have by using (2.3), (2.4) and Lemma 2.1,

$$\int_{-\infty}^{\infty} E_\mu[\tilde{R}(c,\tau^*)]\exp(-\mu^2/2\sigma_0^2)d\mu$$

$$= 2^{1/3}\, 3^{-1}\, c^{2/3} \int_1^{\infty} (z-z^{-1} + 2\ln z)^{-\frac{4}{3}} (1 + \ln z-z^{-1}) \times$$

$$(1 + 2z^{-1} + z^{-2}) \exp[-c^{2/3}(z-z^{-1} + 2\ln z)^{2/3}\sigma_0^{-2}\, 2^{-5/3}]dz$$

Let

$$\gamma = 2^{-5/3}\, c^{2/3}\, \sigma_0^{-2}$$

$$I(z) = (z - z^{-1} + 2 \ln z)^{-4/3}(1 + \ln z - z^{-1})(1 + 2z^{-1} + z^{-2})$$

We have

$$(2.5) \quad \int_{-\infty}^{\infty} E_\mu\{\tilde{R}(c,\tau^*)\}\exp(-\mu^2/2\sigma_0^2)\,d\mu = 2^{1/3}3^{-1}c^{2/3} \;\times$$

$$\int_1^{\infty} I(z)\, \exp(-\gamma(z - z^{-1} + 2 \ln z)^{2/3})\,dz$$

<u>Lemma 2.3</u>: $\int_1^{1/\gamma} I(z)\exp(-\gamma(z-z^{-1} + 2 \ln z)^{2/3})\,dz$

$$= \int_1^{\infty} I(z)\,dz + 3\gamma^{1/3} \ln\gamma - 12\gamma^{1/3} + O(\gamma^{2/3}\ln\gamma).$$

<u>Proof:</u>

$$\int_1^{1/\gamma} I(z)\exp(-\gamma(z-z^{-1} + 2 \ln z)^{2/3})\,dz$$

$$= \int_1^{\gamma^{-1}} I(z)\,[1 - \gamma(z - z^{-1} + 2 \ln z)^{2/3}\,(1 + o(1))]\,dz$$

$$= \int_1^{\gamma^{-1}} I(z)\,dz - \gamma(1 + o(1)) \int_1^{\gamma^{-1}} I(z)(z - z^{-1} + 2 \ln z)^{2/3}\,dz$$

$$= \int_1^{\infty} I(z)\,dz - \int_{\gamma^{-1}}^{\infty} I(z)\,dz - \gamma(1 + o(1))\,O(\gamma^{-1/3} \ln\gamma)$$

$$= \int_1^{\infty} I(z)\,dz + 3\gamma^{1/3} \ln\gamma - 12\gamma^{1/3} + O(\gamma^{2/3} \ln\gamma)$$

<u>Lemma 2.4:</u> $\int_{1/\gamma}^{\infty} I(z)\exp(-\gamma(z - z^{-1} + 2 \ln z)^{2/3})\,dz$

$$= 12\gamma^{1/3} - 3\gamma^{1/3} \ln\gamma + 9 \cdot 2^{-1}\, \pi^{\frac{1}{2}}\, \gamma^{\frac{1}{2}} \ln\gamma\,(1 + o(1))$$

<u>Proof:</u> Let $w = \gamma(z - z^{-1} + 2 \ln z)^{2/3}$

$$I(z)\exp(-\gamma(z - z^{-1} + 2 \ln z)^{2/3})\,dz$$

$$= 3 \cdot 2^{-1}\, \gamma^{1/2}\, w^{-3/2}(1 + \ln z - z^{-1})e^{-w}\,dw$$

Let

$$u = z - z^{-1} + 2 \ln z = (w/\gamma)^{3/2}$$

For $z \geq \gamma^{-1}$

$$1 + \ln z - z^{-1} = 1 + \ln u + O(u^{-1} \ln u)$$

$$= 1 + 3 \cdot 2^{-1} \ln(w/\gamma) + O((w/\gamma)^{-3/2} \ln (w/\gamma))$$

Then

$$\int_{\gamma^{-1}}^{\infty} I(z) \exp(-\gamma(z - z^{-1} + 2 \ln z)^{2/3}) dz$$

$$= 3 \cdot 2^{-2} \int_{\gamma(\gamma^{-1} - \gamma - 2 \ln \gamma)^{2/3}}^{\infty} \gamma^{1/2} w^{-3/2} (3 \ln w - 3 \ln \gamma + 2) e^{-w} dw +$$

$$O(\gamma^{4/3} \ln \gamma)$$

$$= 12\gamma^{1/3} - 3\gamma^{1/3} \ln \gamma + 9 \cdot 2^{-1} \pi^{1/2} \gamma^{1/2} \ln \gamma (1 + o(1))$$

From (2.5), Lemma 2.3 and Lemma 2.4 we get the Theorem.

From (1.1), Lemma 2.2 and the Theorem, we have the following corollary to the Theorem.

__Corollary:__ $K > K'$

__Remark:__ Consider the case of ν having a prior distribution of Lebesgue measure. For any stopping rule τ,

$$\int_{-\infty}^{\infty} R(\nu,\tau) d\nu = \lim_{\sigma_0 \to \infty} (2\pi\sigma_0^2)^{\frac{1}{2}} [(2\pi\sigma_0^2)^{-1/2} \int_{-\infty}^{\infty} R(\nu,\tau) e^{-\nu^2/2\sigma_0^2} d\nu]$$

$$\geq \lim_{\sigma_0 \to \infty} (2\pi\sigma_0^2)^{1/2} B(0,\sigma_0^2)$$

$$= K c^{2/3}$$

So the Bayes risk with respect to Lebesgue measure

$$\inf_{\tau} \int_{-\infty}^{\infty} R(\nu,\tau) d\nu \geq K c^{2/3} > K' c^{2/3}$$

for all $c > 0$.

Here, $K' c^{2/3}$ is the lower bound derived in (1). Therefore, we have shown that Bickel and Yahav's lower bound cannot be attained.

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[3] Chernoff, H. (1965). "Sequential tests for the mean of a normal distribution III (Small t)". Ann. Math. Statist, , 36, 28 - 54.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 26	2. GOVT ACCESSION NO. AD-A127731	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Lower Bound For The Bayes Risk For Testing Sequentially The Sign Of The Drift Parameter Of A Wiener Process		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ashim Mallik and Yi-Ching Yao		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0555
9. PERFORMING ORGANIZATION NAME AND ADDRESS Statistics Center Massachusetts Institute of Technology Cambridge, MA 02139		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (NR-609-001)
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics and Probability Code 436 Arlington, VA 22217		12. REPORT DATE March 1983
		13. NUMBER OF PAGES 9
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Sequential tests, S.P.R.T, Bayes, stopping times, lower bound, asymptotic expansion

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

See reverse side.

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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